

A new S -type eigenvalue localization set for tensors and its applications *

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Abstract

A new S -type eigenvalue localization set for tensors is derived by breaking $N = \{1, 2, \dots, n\}$ into disjoint subsets S and its complement. It is proved that this new set is tighter than those presented by Qi (Journal of Symbolic Computation 40 (2005) 1302-1324), Li et al. (Numer. Linear Algebra Appl. 21 (2014) 39-50) and Li et al. (Linear Algebra Appl. 493 (2016) 469-483). As applications, checkable sufficient conditions for the positive definiteness and the positive semi-definiteness of tensors are proposed. Moreover, based on this new set, we establish a new upper bound for the spectral radius of nonnegative tensors and a lower bound for the minimum H -eigenvalue of weakly irreducible strong M -tensors in this paper. We demonstrate that these bounds are sharper than those obtained by Li et al. (Numer. Linear Algebra Appl. 21 (2014) 39-50) and He and Huang (J. Inequal. Appl. 114 (2014) 2014). Numerical examples are also given to illustrate this fact.

Key Words: Tensor eigenvalue, Localization set, Positive (semi-)definite, Nonnegative tensor, Spectral radius, Nonsingular M -tensors, Minimum H -eigenvalue.

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1 Introduction

Eigenvalue problems of higher order tensors have become an important topic in applied mathematics branch, numerical multilinear algebra, and it has a wide range of practical applications, such as best-rank one approximation in data analysis [1], higher order Markov chains [2], molecular conformation [3] and so forth. Recently, tensor eigenvalues have received much attention in the literatures [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

One of many practical applications of eigenvalues of tensors is that one can use the smallest H -eigenvalue of an even-order real symmetric tensor to identify its positive (semi-)definiteness,

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consequently, can identify the positive (semi-)definiteness of the multivariate homogeneous polynomial determined by this tensor, for details, see [4, 20, 21].

However, as mentioned in [22, 20, 23], it is not easy to compute the smallest H -eigenvalue of tensors when the order and dimension are very large, we always try to give a set including all eigenvalues in the complex. Some sets including all eigenvalues of tensors have been presented by some researchers [4, 19, 20, 21, 22, 23, 24, 25]. In particular, if one of these sets for an even-order real symmetric tensor is in the right-half complex plane, then we can conclude that the smallest H -eigenvalue is positive, consequently, the corresponding tensor is positive definite. Therefore, the main aim of this paper is to study the new eigenvalue localization set for tensors called new S -type eigenvalue localization set, which is sharper than some existing ones.

For a positive integer n , N denotes the set $N = \{1, 2, \dots, n\}$. The set of all real numbers is denoted by \mathbb{R} , and \mathbb{C} denotes the set of all complex numbers. Here, we call $\mathcal{A} = (a_{i_1 \dots i_m})$ a complex (real) tensor of order m dimension n , denoted by $\mathbb{C}^{[m, n]}(\mathbb{R}^{[m, n]})$, if $a_{i_1 \dots i_m} \in \mathbb{C}(\mathbb{R})$, where $i_j \in N$ for $j = 1, 2, \dots, m$ [22].

Let $\mathcal{A} \in \mathbb{R}^{[m, n]}$, and $x \in \mathbb{C}^n$. Then

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n},$$

a pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n / \{0\})$ is called an eigenpair of \mathcal{A} [16] if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1})^T$ [24]. Furthermore, we call (λ, x) an H -eigenpair, if both λ and x are real [4].

A real tensor of order m dimension n is called the unit tensor [20], denoted by \mathcal{I} , if its entries are $\delta_{i_1 \dots i_m}$ for $i_1, \dots, i_m \in N$, where

$$\delta_{i_1 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

An m -order n -dimensional tensor \mathcal{A} is called nonnegative [7, 10, 11, 6, 26], if each entry is nonnegative. We call a tensor \mathcal{A} as a Z -tensor, if all of its off-diagonal entries are non-positive, which is equivalent to write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and \mathcal{B} is a nonnegative tensor ($\mathcal{B} \geq 0$), denote by \mathbb{Z} the set of m -order and n -dimensional Z -tensors. A Z -tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is an M -tensor if $s \geq \rho(\mathcal{B})$, and it is a nonsingular (strong) M -tensor if $s > \rho(\mathcal{B})$ [18, 27, 28].

A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m, n]}$ is called weakly reducible, if there exists a nonempty proper index subset $I \subset N$ such that $a_{i_1 i_2 \dots i_m} = 0$, $\forall i_1 \in I$, $\exists i_j \notin I$, $j = 2, \dots, m$. If \mathcal{A} is not weakly reducible, then we call \mathcal{A} weakly irreducible [11, 19]. The tensor \mathcal{A} is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset N$ such that $a_{i_1 i_2 \dots i_m} = 0$, $\forall i_1 \in \mathbb{J}$, $\forall i_2, \dots, i_m \notin \mathbb{J}$. If \mathcal{A} is not reducible, then we call \mathcal{A} is irreducible [17]. The spectral radius $\rho(\mathcal{A})$ [11] of the tensor \mathcal{A} is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

Denoted by $\tau(\mathcal{A})$ the minimum value of the real part of all eigenvalues of the strong M -tensor \mathcal{A} [12]. A real tensor $\mathcal{A} = (a_{i_1 \dots i_m})$ is called symmetric [4, 10, 19, 22, 25, 21] if

$$a_{i_1 \dots i_m} = a_{\pi(i_1 \dots i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices.

Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$. For $i, j \in N$, $j \neq i$, denote

$$\begin{aligned} R_i(\mathcal{A}) &= \sum_{i_2, \dots, i_m=1}^n a_{ii_2 \dots i_m}, \quad R_{\max}(\mathcal{A}) = \max_{i \in N} R_i(\mathcal{A}), \quad R_{\min}(\mathcal{A}) = \min_{i \in N} R_i(\mathcal{A}), \\ r_i(\mathcal{A}) &= \sum_{\delta_{ii_2 \dots i_m}=0} |a_{ii_2 \dots i_m}|, \quad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2 \dots i_m}=0, \\ \delta_{ji_2 \dots i_m}=0}} |a_{ii_2 \dots i_m}| = r_i(\mathcal{A}) - |a_{ij \dots j}|. \end{aligned}$$

Recently, many literatures have been focused on the bounds of the spectral radius of nonnegative tensor in [11, 13, 14, 15, 16, 17, 29]. Also, in [12], He and Huang obtained the upper and lower bounds for the minimum H -eigenvalue of irreducible strong M -tensors. Wang and Wei [14] presented some new bounds for the minimum H -eigenvalue of weakly irreducible strong M -tensors, and showed those are better than the ones in [12] in some cases. Based on the new set established in this paper, the other main results of this paper is to provide sharper bounds for the spectral radius of nonnegative tensors and the minimum H -eigenvalue of weakly irreducible nonsingular M -tensors, which improve some existing ones.

Before presenting our results, we review the existing results related to the eigenvalue localization sets for tensors. In 2005, Qi [12] generalized Geršgorin eigenvalue localization theorem from matrices to real supersymmetric tensors, which can be easily extended to general tensors [10, 25].

Lemma 1.1. [4] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m, n]}$, $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}),$$

where $\sigma(\mathcal{A})$ is the set of all the eigenvalues of \mathcal{A} and

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i \dots i}| \leq r_i(\mathcal{A})\}.$$

To get sharper eigenvalue localization sets than $\Gamma(\mathcal{A})$, Li et al. [25] extended the Brauer's eigenvalue localization set of matrices [30] and proposed the following Brauer-type eigenvalue localization set for tensors.

Lemma 1.2. [25] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m, n]}$, $n \geq 2$. Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) = \bigcup_{i, j \in N, j \neq i} \mathcal{K}_{i, j}(\mathcal{A}),$$

where

$$\mathcal{K}_{i, j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i \dots i}| - r_i^j(\mathcal{A}))|z - a_{j \dots j}| \leq |a_{ij \dots j}|r_j(\mathcal{A})\}.$$

In addition, in order to reduce computations of determining the sets $\sigma(\mathcal{A})$, Li et al. [25] also presented the following S -type eigenvalue localization set by breaking N into disjoint subsets S and \bar{S} , where \bar{S} is the complement of S in N .

Lemma 1.3. [25] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m, n]}$, $n \geq 2$, and S be a nonempty proper subset of N . Then*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = \left(\bigcup_{i \in S, j \in \bar{S}} \mathcal{K}_{i, j}(\mathcal{A}) \right) \cup \left(\bigcup_{i \in \bar{S}, j \in S} \mathcal{K}_{i, j}(\mathcal{A}) \right),$$

where $\mathcal{K}_{i, j}(\mathcal{A})$ ($i \in S, j \in \bar{S}$ or $i \in \bar{S}, j \in S$) is defined as in Lemma 1.2.

Very recently, by the technique in [25], Li et al. [22] gave the new eigenvalue localization set involved with a proper subset S of N , and by the following three sets:

$$\begin{aligned}\Delta^N &= \{(i_2, i_3, \dots, i_m) : \text{each } i_j \in N \text{ for } j = 2, 3, \dots, m\}, \\ \Delta^S &= \{(i_2, i_3, \dots, i_m) : \text{each } i_j \in S \text{ for } j = 2, 3, \dots, m\}, \quad \overline{\Delta^S} = \Delta^N \setminus \Delta^S.\end{aligned}$$

Lemma 1.4. [22] *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m, n]}$, $n \geq 2$, and S be a nonempty proper subset of N . Then*

$$\sigma(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) = \left(\bigcup_{i \in S, j \in \bar{S}} \Omega_{i,j}^S(\mathcal{A}) \right) \cup \left(\bigcup_{i \in \bar{S}, j \in S} \Omega_{i,j}^{\bar{S}}(\mathcal{A}) \right),$$

where

$$\begin{aligned}\Omega_{i,j}^S(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i \dots i}|)(|z - a_{j \dots j}| - r_j^{\bar{S}}(\mathcal{A})) \leq r_i(\mathcal{A})r_j^{\Delta^S}(\mathcal{A})\}, \\ \Omega_{i,j}^{\bar{S}}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i \dots i}|)(|z - a_{j \dots j}| - r_j^{\Delta^S}(\mathcal{A})) \leq r_i(\mathcal{A})r_j^{\bar{S}}(\mathcal{A})\},\end{aligned}$$

and for $i \in S$,

$$r_i(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\bar{S}}(\mathcal{A}), \quad r_i^j(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\bar{S}}(\mathcal{A}) - |a_{ij \dots j}|,$$

with

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2, \dots, i_m) \in \Delta^S, \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad r_i^{\bar{S}}(\mathcal{A}) = \sum_{(i_2, \dots, i_m) \in \bar{\Delta^S}} |a_{ii_2 \dots i_m}|.$$

Theorem 6 in [22] shows that this new set is tighter than the sets $\Gamma(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}^S(\mathcal{A})$.

In this paper, we focus on investigating the eigenvalue localization sets for tensors, and obtain a new S -type eigenvalue localization set for tensors. It is proved to be tighter than the tensor Geršgorin eigenvalue localization set $\Gamma(\mathcal{A})$ in Lemma 1.1, the Brauer's eigenvalue localization set $\mathcal{K}(\mathcal{A})$ in Lemma 1.2, the S -type eigenvalue localization set $\mathcal{K}^S(\mathcal{A})$ in Lemma 1.3 and another S -type eigenvalue localization set $\Omega^S(\mathcal{A})$ in Lemma 1.4. As applications, checkable sufficient conditions for the positive definiteness and the positive semi-definiteness of tensors are proposed, and some new bounds for the spectral radius of nonnegative tensors and the minimum H -eigenvalue of weakly irreducible strong M -tensors are established. The bounds improve some existing ones. Numerical examples are implemented to illustrate this fact.

The outline of this paper is organized as follows. In Section 2, we recollect some useful lemmas which are utilized in the next sections. In Section 3, a new S -type eigenvalue localization set for tensors is given, and proved to be tighter than the existing ones derived in Lemmas 1.1-1.4. As applications of the results in Section 3, checkable sufficient conditions for the positive definiteness and the positive semi-definiteness of tensors are given in Section 4. Based on the results of Section 3, we propose a new upper bound for the spectral radius of nonnegative tensors in Section 5, comparison results for this new bound and those derived in [25] are also investigated in this section. Section 6 is devoted to exhibit a new lower bound for the minimum H -eigenvalue of weakly irreducible strong M -tensors, which is proved to be sharper than the ones obtained by He and Huang [12]. Finally, some concluding remarks are given to end this paper in Section 7.

2 Preliminaries

In this section, we start with some lemmas. They will be useful in the following proofs.

Lemma 2.1. [10] *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is nonnegative, then $\rho(\mathcal{A})$ is an eigenvalue with an entrywise nonnegative eigenvector x , i.e., $x \geq 0$, $x \neq 0$, corresponding to it.*

Lemma 2.2. [25] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor. Then $\rho(\mathcal{A}) \geq \max_{i \in N} \{a_{i \dots i}\}$.*

Lemma 2.3. [12] *Let \mathcal{A} be a strong M -tensor and denoted by $\tau(\mathcal{A})$ the minimum value of the real part of all eigenvalues of \mathcal{A} . Then $\tau(\mathcal{A}) > 0$ is an eigenvalue of \mathcal{A} with a nonnegative eigenvector. Moreover, if \mathcal{A} is irreducible, then $\tau(\mathcal{A})$ is a unique eigenvalue with a positive eigenvector.*

Lemma 2.4. [14] *Let \mathcal{A} be a weakly irreducible strong M -tensor. Then $\tau(\mathcal{A}) \leq \min_{i \in N} \{a_{i \dots i}\}$.*

Lemma 2.5. [22] *Let $a, b, c \geq 0$ and $d > 0$.*

(I) *If $\frac{a}{b+c+d} \leq 1$, then*

$$\frac{a - (b + c)}{d} \leq \frac{a - b}{c + d} \leq \frac{a}{b + c + d}.$$

(II) *If $\frac{a}{b+c+d} \geq 1$, then*

$$\frac{a - (b + c)}{d} \geq \frac{a - b}{c + d} \geq \frac{a}{b + c + d}.$$

3 A new S -type eigenvalue localization set for tensors

In this section, we investigate eigenvalue localization sets and present a new S -type eigenvalue localization set for tensors, and the comparison results of this new set with those in Lemmas 1.1-1.4 are established.

Theorem 3.1. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$ and S be a nonempty proper subset of N . Then*

$$\sigma(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A}) := \left(\Upsilon_{i,j}^S(\mathcal{A}) \right) \cup \left(\Upsilon_{i,j}^{\bar{S}}(\mathcal{A}) \right), \quad (1)$$

where

$$\begin{aligned} \Upsilon_{i,j}^S(\mathcal{A}) &= \left(\bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A}) \right) \cup \left(\bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right), \\ \Upsilon_{i,j}^{\bar{S}}(\mathcal{A}) &= \left(\bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A}) \right) \cup \left(\bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}_{i,j}^2(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right), \end{aligned}$$

with

$$\begin{aligned} \hat{\Upsilon}_i^1(\mathcal{A}) &= \{z \in \mathbb{C} : |z - a_{i \dots i}| \leq r_i^{\overline{\Delta \bar{S}}}(\mathcal{A})\}, \\ \hat{\Upsilon}_i^2(\mathcal{A}) &= \{z \in \mathbb{C} : |z - a_{i \dots i}| \leq r_i^{\overline{\Delta S}}(\mathcal{A})\}, \\ \tilde{\Upsilon}_{i,j}^1(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i \dots i}| - r_i^{\overline{\Delta \bar{S}}}(\mathcal{A}))(|z - a_{j \dots j}| - r_j^{\Delta \bar{S}}(\mathcal{A})) \leq r_i^{\Delta \bar{S}}(\mathcal{A}) r_j^{\overline{\Delta \bar{S}}}(\mathcal{A})\}, \\ \tilde{\Upsilon}_{i,j}^2(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i \dots i}| - r_i^{\overline{\Delta S}}(\mathcal{A}))(|z - a_{j \dots j}| - r_j^{\Delta S}(\mathcal{A})) \leq r_i^{\Delta S}(\mathcal{A}) r_j^{\overline{\Delta S}}(\mathcal{A})\}. \end{aligned}$$

Proof. For any $\lambda \in \sigma(\mathcal{A})$, let $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n \setminus 0$ be an associated eigenvector, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \quad (2)$$

Let $|x_p| = \max_{i \in S} \{|x_i|\}$ and $|x_q| = \max_{i \in \bar{S}} \{|x_i|\}$. Then, $x_p \neq 0$ or $x_q \neq 0$. Now, let us distinguish two cases to prove.

(i) $|x_p| \geq |x_q|$, so $|x_p| = \max_{i \in N} \{|x_i|\}$ and $|x_p| > 0$. It follows from (2) that

$$\sum_{i_2, \dots, i_m=1}^n a_{pi_2 \dots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_p^{m-1}.$$

Hence, we have

$$(\lambda - a_{p \dots p})x_p^{m-1} = \sum_{\substack{(i_2, \dots, i_m) \in \bar{\Delta}^{\bar{S}}, \\ \delta_{pi_2 \dots i_m} = 0}} a_{pi_2 \dots i_m} x_{i_2} \cdots x_{i_m} + \sum_{(i_2, \dots, i_m) \in \Delta^{\bar{S}}} a_{pi_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking absolute values in the above equation and using the triangle inequality yield

$$\begin{aligned} & |\lambda - a_{p \dots p}| |x_p|^{m-1} \\ & \leq \sum_{\substack{(i_2, \dots, i_m) \in \bar{\Delta}^{\bar{S}}, \\ \delta_{pi_2 \dots i_m} = 0}} |a_{pi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2, \dots, i_m) \in \Delta^{\bar{S}}} |a_{pi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\ & \leq \sum_{\substack{(i_2, \dots, i_m) \in \bar{\Delta}^{\bar{S}}, \\ \delta_{pi_2 \dots i_m} = 0}} |a_{pi_2 \dots i_m}| |x_p|^{m-1} + \sum_{(i_2, \dots, i_m) \in \Delta^{\bar{S}}} |a_{pi_2 \dots i_m}| |x_q|^{m-1} \\ & = r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) |x_p|^{m-1} + r_p^{\Delta^{\bar{S}}}(\mathcal{A}) |x_q|^{m-1}, \end{aligned}$$

which means that

$$(|\lambda - a_{p \dots p}| - r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})) |x_p|^{m-1} \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A}) |x_q|^{m-1}. \quad (3)$$

If $|x_q| = 0$, it follows from (3) that $|\lambda - a_{p \dots p}| - r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) \leq 0$ by $|x_p| > 0$, that is, $|\lambda - a_{p \dots p}| \leq r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})$. Evidently, $\lambda \in \hat{\Upsilon}_p^1(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. Otherwise, $|x_q| > 0$. If $\lambda \notin \bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A})$, it is easy to see that for any $i \in S$,

$$|\lambda - a_{i \dots i}| > r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}).$$

In particular, $|\lambda - a_{p \dots p}| > r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})$, i.e., $|\lambda - a_{p \dots p}| - r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) > 0$. By (3), it is not difficult to verify that $\lambda \in \Gamma_p(\mathcal{A})$. Besides, it follows from (2) that

$$\begin{aligned} & |\lambda - a_{q \dots q}| |x_q|^{m-1} \\ & \leq \sum_{(i_2, \dots, i_m) \in \bar{\Delta}^{\bar{S}}} |a_{qi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{\substack{(i_2, \dots, i_m) \in \Delta^{\bar{S}}, \\ \delta_{qi_2 \dots i_m} = 0}} |a_{qi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\ & \leq \sum_{(i_2, \dots, i_m) \in \bar{\Delta}^{\bar{S}}} |a_{qi_2 \dots i_m}| |x_p|^{m-1} + \sum_{\substack{(i_2, \dots, i_m) \in \Delta^{\bar{S}}, \\ \delta_{qi_2 \dots i_m} = 0}} |a_{qi_2 \dots i_m}| |x_q|^{m-1} \\ & = r_q^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) |x_p|^{m-1} + r_q^{\Delta^{\bar{S}}}(\mathcal{A}) |x_q|^{m-1}, \end{aligned}$$

which is equivalent to

$$(|\lambda - a_{q\dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_q|^{m-1} \leq r_q^{\Delta^{\bar{S}}}(\mathcal{A})|x_p|^{m-1}. \quad (4)$$

Note that $|x_p| > 0$ and $|\lambda - a_{p\dots p}| > r_p^{\Delta^{\bar{S}}}(\mathcal{A})$, multiplying (3) with (4) results in

$$\begin{aligned} & (|\lambda - a_{p\dots p}| - r_p^{\Delta^{\bar{S}}}(\mathcal{A}))(|\lambda - a_{q\dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_q|^{m-1}|x_p|^{m-1} \\ & \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\Delta^{\bar{S}}}(\mathcal{A})|x_p|^{m-1}|x_q|^{m-1}, \end{aligned}$$

which implies that

$$(|\lambda - a_{p\dots p}| - r_p^{\Delta^{\bar{S}}}(\mathcal{A}))(|\lambda - a_{q\dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\Delta^{\bar{S}}}(\mathcal{A})$$

by $|x_p| \geq |x_q| > 0$. Therefore, $\lambda \in (\tilde{\Upsilon}_{p,q}^1(\mathcal{A}) \cap \Gamma_p(\mathcal{A})) \subseteq \Upsilon^S(\mathcal{A})$.

(ii) $|x_p| \leq |x_q|$, so $|x_q| = \max_{i \in N} \{|x_i|\}$ and $|x_q| > 0$. It follows from (2) that

$$\sum_{i_2, \dots, i_m=1}^n a_{qi_2\dots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_q^{m-1}.$$

Therefore, we have

$$(\lambda - a_{q\dots q})x_q^{m-1} = \sum_{(i_2, \dots, i_m) \in \Delta^S} a_{qi_2\dots i_m} x_{i_2} \cdots x_{i_m} + \sum_{\substack{(i_2, \dots, i_m) \in \overline{\Delta^S}, \\ \delta_{qi_2\dots i_m} = 0}} a_{qi_2\dots i_m} x_{i_2} \cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned} & |\lambda - a_{q\dots q}||x_q|^{m-1} \\ & \leq \sum_{(i_2, \dots, i_m) \in \Delta^S} |a_{qi_2\dots i_m}||x_{i_2}| \cdots |x_{i_m}| + \sum_{\substack{(i_2, \dots, i_m) \in \overline{\Delta^S}, \\ \delta_{qi_2\dots i_m} = 0}} |a_{qi_2\dots i_m}||x_{i_2}| \cdots |x_{i_m}| \\ & \leq \sum_{(i_2, \dots, i_m) \in \Delta^S} |a_{qi_2\dots i_m}||x_p|^{m-1} + \sum_{\substack{(i_2, \dots, i_m) \in \overline{\Delta^S}, \\ \delta_{qi_2\dots i_m} = 0}} |a_{qi_2\dots i_m}||x_q|^{m-1} \\ & = r_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1} + r_q^{\Delta^{\bar{S}}}(\mathcal{A})|x_q|^{m-1}, \end{aligned}$$

which yields that

$$(|\lambda - a_{q\dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))|x_q|^{m-1} \leq r_q^{\Delta^S}(\mathcal{A})|x_p|^{m-1}. \quad (5)$$

If $|x_p| = 0$, it follows from (5) that $|\lambda - a_{q\dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}) \leq 0$ by $|x_q| > 0$, i.e., $|\lambda - a_{q\dots q}| \leq r_q^{\Delta^{\bar{S}}}(\mathcal{A})$, obviously, $\lambda \in \hat{\Upsilon}_q^2(\mathcal{A}) \subseteq \Upsilon^S(\mathcal{A})$. Otherwise, $|x_p| > 0$. If $\lambda \notin \bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A})$, we are easy to see that for any $i \in \bar{S}$,

$$|\lambda - a_{i\dots i}| > r_i^{\Delta^{\bar{S}}}(\mathcal{A}).$$

In particular, $|\lambda - a_{q\dots q}| > r_q^{\overline{\Delta^S}}(\mathcal{A})$, i.e., $|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A}) > 0$. By (5), we infer that $\lambda \in \Gamma_q(\mathcal{A})$. In addition, it follows from (2) that

$$\begin{aligned}
& |\lambda - a_{p\dots p}| |x_p|^{m-1} \\
& \leq \sum_{\substack{(i_2, \dots, i_m) \in \Delta^S, \\ \delta_{pi_2 \dots i_m} = 0}} |a_{pi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{(i_2, \dots, i_m) \in \overline{\Delta^S}} |a_{pi_2 \dots i_m}| |x_{i_2}| \cdots |x_{i_m}| \\
& \leq \sum_{\substack{(i_2, \dots, i_m) \in \Delta^S, \\ \delta_{pi_2 \dots i_m} = 0}} |a_{pi_2 \dots i_m}| |x_p|^{m-1} + \sum_{(i_2, \dots, i_m) \in \overline{\Delta^S}} |a_{pi_2 \dots i_m}| |x_q|^{m-1} \\
& = r_p^{\Delta^S}(\mathcal{A}) |x_p|^{m-1} + r_p^{\overline{\Delta^S}}(\mathcal{A}) |x_q|^{m-1},
\end{aligned}$$

which is equivalent to

$$(|\lambda - a_{p\dots p}| - r_p^{\Delta^S}(\mathcal{A})) |x_p|^{m-1} \leq r_p^{\overline{\Delta^S}}(\mathcal{A}) |x_q|^{m-1}. \quad (6)$$

Having in mind that $|x_q| > 0$ and $|\lambda - a_{q\dots q}| > r_q^{\overline{\Delta^S}}(\mathcal{A})$, multiplying (5) with (6) results in

$$\begin{aligned}
& (|\lambda - a_{p\dots p}| - r_p^{\Delta^S}(\mathcal{A})) (|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})) |x_q|^{m-1} |x_p|^{m-1} \\
& \leq r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A}) |x_p|^{m-1} |x_q|^{m-1},
\end{aligned}$$

which results in

$$(|\lambda - a_{p\dots p}| - r_p^{\Delta^S}(\mathcal{A})) (|\lambda - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})) \leq r_p^{\overline{\Delta^S}}(\mathcal{A}) r_q^{\Delta^S}(\mathcal{A})$$

by $|x_q| \geq |x_p| > 0$. This leads to $\lambda \in (\tilde{\Upsilon}_{q,p}^2(\mathcal{A}) \cap \Gamma_q(\mathcal{A})) \subseteq \Upsilon^S(\mathcal{A})$. This completes our proof of Theorem 3.1. \blacksquare

Now, we establish a comparison result between $\Upsilon^S(\mathcal{A})$, $\Omega^S(\mathcal{A})$, $\mathcal{K}^S(\mathcal{A})$, $\mathcal{K}(\mathcal{A})$ and $\Gamma(\mathcal{A})$ as follows.

Theorem 3.2. *Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m,n]}$, $n \geq 2$ and S be a nonempty proper subset of N . Then*

$$\Upsilon^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. By Theorem 6 in [22], we see that $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ holds. Thus, we only need to prove $\Upsilon^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. Let $z \in \Upsilon^S(\mathcal{A})$. Then

$$z \in \Upsilon_{i,j}^S(\mathcal{A}) \text{ or } z \in \Upsilon_{i,j}^{\bar{S}}(\mathcal{A}).$$

Without loss of generality, we first assume that $z \in \Upsilon_{i,j}^S(\mathcal{A})$. If $z \in \bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A})$, then there exists one index $i_0 \in S$ such that

$$|z - a_{i_0 \dots i_0}| \leq r_{i_0}^{\overline{\Delta^S}}(\mathcal{A}),$$

i.e., $|z - a_{i_0 \dots i_0}| - r_{i_0}^{\overline{\Delta^S}}(\mathcal{A}) \leq 0$. Hence, for any $i \in \bar{S}$, it follows

$$(|z - a_{i \dots i}|) (|z - a_{i_0 \dots i_0}| - r_{i_0}^{\overline{\Delta^S}}(\mathcal{A})) \leq r_i(\mathcal{A}) r_{i_0}^{\Delta^{\bar{S}}}(\mathcal{A}),$$

which implies that $z \in \Omega_{i,i_0}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. Otherwise, $z \notin \bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A})$, then

$$z \in \left(\bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right) \quad (7)$$

and

$$|z - a_{i \dots i}| > r_i^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \quad (8)$$

for any $i \in S$. It follows from (7) that there exist $p \in S$ and $q \in \bar{S}$ such that

$$|z - a_{p \dots p}| \leq r_p(\mathcal{A}) \quad (9)$$

and

$$(|z - a_{p \dots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|z - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}). \quad (10)$$

If $r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) = 0$, combining (8) and (10) results in

$$|z - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}) \leq 0 \leq r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}), \quad (11)$$

that is, $|z - a_{q \dots q}| \leq r_q(\mathcal{A})$, which is equivalent to

$$|z - a_{q \dots q}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) \leq r_q^{\Delta^S}(\mathcal{A}). \quad (12)$$

Multiplying (9) with (12) yields

$$(|z - a_{p \dots p}|)(|z - a_{q \dots q}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq r_p(\mathcal{A})r_q^{\Delta^S}(\mathcal{A}). \quad (13)$$

This means that $z \in \Omega_{p,q}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

In the sequel, we discuss the case $r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) > 0$, then by dividing (10) by $r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})$ is given by

$$\frac{(|z - a_{p \dots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))(|z - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A}))}{r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})} \leq 1. \quad (14)$$

If $\frac{|z - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})}{r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})} \geq 1$, let $a = |z - a_{q \dots q}| \geq 0$, $b + c = r_q^{\Delta^{\bar{S}}}(\mathcal{A}) \geq 0$ with $b, c \geq 0$ and $d = r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}) > 0$, then by (II) of Lemma 2.5, we have

$$\frac{|z - a_{p \dots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})}{r_p^{\Delta^{\bar{S}}}(\mathcal{A})} \frac{|z - a_{q \dots q}|}{r_q(\mathcal{A})} \leq \frac{|z - a_{p \dots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})}{r_p^{\Delta^{\bar{S}}}(\mathcal{A})} \frac{|z - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})}{r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})} \leq 1,$$

which is equivalent to

$$(|z - a_{q \dots q}|)(|z - a_{p \dots p}| - r_p^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq r_q(\mathcal{A})r_p^{\Delta^{\bar{S}}}(\mathcal{A}). \quad (15)$$

This implies that $z \in \Omega_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. Furthermore, if $\frac{|z - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})}{r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})} \leq 1$, then (12) holds.

Multiplying (9) with (12) leads to

$$(|z - a_{p \dots p}|)(|z - a_{q \dots q}| - r_q^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq r_p(\mathcal{A})r_q^{\Delta^S}(\mathcal{A}), \quad (16)$$

which implies that $z \in \Omega_{p,q}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

On the other hand, we prove the case $z \in \Upsilon_{i,j}^{\bar{S}}(\mathcal{A})$. If $z \in \bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A})$, then there is one index $i_1 \in \bar{S}$ such that

$$|z - a_{i_1 \dots i_1}| \leq r_{i_1}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}),$$

i.e., $|z - a_{i_1 \dots i_1}| - r_{i_1}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) \leq 0$. Then, for any $i \in S$, we deduce that

$$(|z - a_{i_1 \dots i_1}|)(|z - a_{i_1 \dots i_1}| - r_{i_1}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})) \leq r_i(\mathcal{A})r_{i_1}^{\Delta^S}(\mathcal{A}),$$

which means that $z \in \Omega_{i,i_1}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. In addition, $z \notin \bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A})$, then

$$z \in \left(\bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}_{i,j}^2(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right) \quad (17)$$

and

$$|z - a_{j \dots j}| > r_j^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) \quad (18)$$

for any $j \in \bar{S}$. It follows from (17) that there exist $p \in S$ and $q \in \bar{S}$ such that

$$|z - a_{q \dots q}| \leq r_q(\mathcal{A}) \quad (19)$$

and

$$(|z - a_{q \dots q}| - r_q^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}))(|z - a_{p \dots p}| - r_p^{\Delta^S}(\mathcal{A})) \leq r_q^{\Delta^S}(\mathcal{A})r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}). \quad (20)$$

If $r_q^{\Delta^S}(\mathcal{A})r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) = 0$, combining (18) and (20) results in

$$|z - a_{p \dots p}| - r_p^{\Delta^S}(\mathcal{A}) \leq 0 \leq r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}), \quad (21)$$

which leads to $|z - a_{p \dots p}| \leq r_p(\mathcal{A})$, and therefore

$$|z - a_{p \dots p}| - r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A}). \quad (22)$$

Multiplying (19) with (22) derives

$$(|z - a_{q \dots q}|)(|z - a_{p \dots p}| - r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})) \leq r_q(\mathcal{A})r_p^{\Delta^{\bar{S}}}(\mathcal{A}). \quad (23)$$

It follows from (23) that $z \in \Omega_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

Afterwards, we investigate the case $r_q^{\Delta^S}(\mathcal{A})r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) > 0$, then by dividing (20) by $r_q^{\Delta^S}(\mathcal{A})r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})$, we have

$$\frac{(|z - a_{q \dots q}| - r_q^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}))(|z - a_{p \dots p}| - r_p^{\Delta^S}(\mathcal{A}))}{r_q^{\Delta^S}(\mathcal{A})r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})} \leq 1. \quad (24)$$

If $\frac{|z - a_{p \dots p}| - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})} \geq 1$, let $a = |z - a_{p \dots p}| \geq 0$, $b + c = r_p^{\Delta^S}(\mathcal{A}) \geq 0$ with $b, c \geq 0$ and $d = r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) > 0$, then by (II) of Lemma 2.5, we have

$$\frac{|z - a_{q \dots q}| - r_q^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \frac{|z - a_{p \dots p}|}{r_p(\mathcal{A})} \leq \frac{|z - a_{q \dots q}| - r_q^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})}{r_q^{\Delta^S}(\mathcal{A})} \frac{|z - a_{p \dots p}| - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})} \leq 1,$$

equivalently,

$$(|z - a_{p\dots p}|)(|z - a_{q\dots q}| - r_q^{\overline{\Delta^S}}(\mathcal{A})) \leq r_p(\mathcal{A})r_q^{\Delta^S}(\mathcal{A}). \quad (25)$$

This implies that $z \in \Omega_{p,q}^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. Furthermore, if $\frac{|z - a_{p\dots p}| - r_p^{\Delta^S}(\mathcal{A})}{r_p^{\overline{\Delta^S}}(\mathcal{A})} \leq 1$, then (22) holds.

Multiplying (19) with (22) leads to

$$(|z - a_{q\dots q}|)(|z - a_{p\dots p}| - r_q^{\overline{\Delta^S}}(\mathcal{A})) \leq r_q(\mathcal{A})r_p^{\Delta^S}(\mathcal{A}), \quad (26)$$

which implies that $z \in \Omega_{q,p}^{\bar{S}}(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

It follows from the above discussions that $\Upsilon^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$. The conclusion follows immediately from what we have proved. \blacksquare

Remark 3.1. For a complex tensor $\mathcal{A} \in \mathbb{C}^{[m,n]}$, $n \geq 2$, the set $\Omega^S(\mathcal{A})$ consists of $|S|(n - |S|)$ sets $\Omega_{i,j}^S(\mathcal{A})$ and $|S|(n - |S|)$ sets $\Omega_{i,j}^{\bar{S}}(\mathcal{A})$, where S is a nonempty proper subset of N , and therefore $\Omega^S(\mathcal{A})$ contains $2|S|(n - |S|)$ sets. In addition, the set $\Upsilon^S(\mathcal{A})$ consists of $|S|(n - |S|)$ sets $\tilde{\Upsilon}_{i,j}^1(\mathcal{A})$, $|S|(n - |S|)$ sets $\tilde{\Upsilon}_{i,j}^2(\mathcal{A})$, $|S|$ sets $\tilde{\Upsilon}_i^1(\mathcal{A})$, $n - |S|$ sets $\tilde{\Upsilon}_i^2(\mathcal{A})$ and n sets $\tilde{\Upsilon}_i(\mathcal{A})$, then there are $2|S|(n - |S|) + 2n$ sets contained in $\Upsilon^S(\mathcal{A})$. Hence there are more computations to determine $\Upsilon^S(\mathcal{A})$ than $\Omega^S(\mathcal{A})$, while $\Upsilon^S(\mathcal{A})$ can capture all eigenvalues of \mathcal{A} more precisely than $\Omega^S(\mathcal{A})$ as showed in Theorem 3.2.

Based on the above discussions, how to choose S to make $\Upsilon^S(\mathcal{A})$ as sharp as possible is very interesting and important. However, this work is difficult especially the dimension of the tensor \mathcal{A} is large. At present, it is very difficult for us to research this problem, we will continue to study this problem in the future.

4 Sufficient conditions for positive (semi-)definiteness of tensors

As applications of the results in Section 3, we provide some checkable sufficient conditions for the positive definiteness and positive semi-definiteness of tensors, respectively in this section. Furthermore, a numerical example is implemented to illustrate the superiority of these conditions to those derived in [22, 20, 25].

Theorem 4.1. Let $\mathcal{A} = (a_{i_1\dots i_m}) \in \mathbb{R}^{[m,n]}$ be an even-order symmetric tensor with $a_{k\dots k} > 0$ for all $k \in N$. If there is a nonempty proper subset S of N and the following four statements hold:

- (i) $a_{i\dots i} > r_i^{\overline{\Delta^S}}(\mathcal{A})$ for any $i \in S$;
- (ii) $a_{i\dots i} > r_i^{\Delta^S}(\mathcal{A})$ for any $i \in \bar{S}$;
- (iii) For any $i \in S$, $j \in \bar{S}$,

$$(a_{i\dots i} - r_i^{\overline{\Delta^S}}(\mathcal{A}))(a_{j\dots j} - r_j^{\Delta^S}(\mathcal{A})) > r_i^{\Delta^S}(\mathcal{A})r_j^{\overline{\Delta^S}}(\mathcal{A})$$

or $a_{i\dots i} > r_i(\mathcal{A})$;

- (iv) For any $i \in \bar{S}$, $j \in S$,

$$(a_{i\dots i} - r_i^{\overline{\Delta^S}}(\mathcal{A}))(a_{j\dots j} - r_j^{\Delta^S}(\mathcal{A})) > r_i^{\Delta^S}(\mathcal{A})r_j^{\overline{\Delta^S}}(\mathcal{A})$$

or $a_{i\dots i} > r_i(\mathcal{A})$, then \mathcal{A} is positive definite.

Proof. Let λ be an H -eigenvalue of \mathcal{A} . We prove this theorem by assuming that $\lambda \leq 0$ and leading a contradiction. From Theorem 3.1, we have $\lambda \in \Upsilon^S(\mathcal{A})$, which implies that there are $i_0, i_1, i_2 \in S$ and $j_0, j_1, j_2 \in \bar{S}$ such that

$$|\lambda - a_{i_0 \dots i_0}| \leq r_{i_0}^{\overline{\Delta^S}}(\mathcal{A}) \text{ or } |\lambda - a_{j_0 \dots j_0}| \leq r_{j_0}^{\overline{\Delta^S}}(\mathcal{A})$$

or

$$(|\lambda - a_{i_1 \dots i_1}| - r_{i_1}^{\overline{\Delta^S}}(\mathcal{A}))(|\lambda - a_{j_1 \dots j_1}| - r_{j_1}^{\Delta^S}(\mathcal{A})) \leq r_{i_1}^{\Delta^S}(\mathcal{A})r_{j_1}^{\overline{\Delta^S}}(\mathcal{A}),$$

$$|\lambda - a_{i_1 \dots i_1}| \leq r_{i_1}(\mathcal{A})$$

or

$$(|\lambda - a_{j_2 \dots j_2}| - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}))(|\lambda - a_{i_2 \dots i_2}| - r_{i_2}^{\Delta^S}(\mathcal{A})) \leq r_{j_2}^{\Delta^S}(\mathcal{A})r_{i_2}^{\overline{\Delta^S}}(\mathcal{A}),$$

$$|\lambda - a_{j_2 \dots j_2}| \leq r_{j_2}(\mathcal{A}).$$

It follows from $a_{k \dots k} > 0$ for all $k \in N$ that

$$|\lambda - a_{i_0 \dots i_0}| \geq a_{i_0 \dots i_0} > r_{i_0}^{\overline{\Delta^S}}(\mathcal{A}) \text{ and } |\lambda - a_{j_0 \dots j_0}| \geq a_{j_0 \dots j_0} > r_{j_0}^{\overline{\Delta^S}}(\mathcal{A})$$

and

$$(|\lambda - a_{i_1 \dots i_1}| - r_{i_1}^{\overline{\Delta^S}}(\mathcal{A}))(|\lambda - a_{j_1 \dots j_1}| - r_{j_1}^{\Delta^S}(\mathcal{A}))$$

$$\geq (a_{i_1 \dots i_1} - r_{i_1}^{\overline{\Delta^S}}(\mathcal{A}))(a_{j_1 \dots j_1} - r_{j_1}^{\Delta^S}(\mathcal{A})) > r_{i_1}^{\Delta^S}(\mathcal{A})r_{j_1}^{\overline{\Delta^S}}(\mathcal{A})$$

or $|\lambda - a_{i_1 \dots i_1}| \geq a_{i_1 \dots i_1} > r_{i_1}(\mathcal{A})$; and

$$(|\lambda - a_{j_2 \dots j_2}| - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}))(|\lambda - a_{i_2 \dots i_2}| - r_{i_2}^{\Delta^S}(\mathcal{A}))$$

$$\geq (a_{j_2 \dots j_2} - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}))(a_{i_2 \dots i_2} - r_{i_2}^{\Delta^S}(\mathcal{A})) > r_{j_2}^{\Delta^S}(\mathcal{A})r_{i_2}^{\overline{\Delta^S}}(\mathcal{A})$$

or $|\lambda - a_{j_2 \dots j_2}| \geq a_{j_2 \dots j_2} > r_{j_2}(\mathcal{A})$. These lead to a contradiction. Hence, $\lambda > 0$, and \mathcal{A} is positive definite. \blacksquare

With the similar manner applied in the proof of Theorem 4.1, we can prove that \mathcal{A} is positive semi-definite in the following theorem.

Theorem 4.2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m, n]}$ be an even-order symmetric tensor with $a_{k \dots k} \geq 0$ for all $k \in N$. If there is a nonempty proper subset S of N and the following four statements hold:

- (i) $a_{i \dots i} \geq r_i^{\overline{\Delta^S}}(\mathcal{A})$ for any $i \in S$;
- (ii) $a_{i \dots i} \geq r_i^{\Delta^S}(\mathcal{A})$ for any $i \in \bar{S}$;
- (iii) For any $i \in S, j \in \bar{S}$,

$$(a_{i \dots i} - r_i^{\overline{\Delta^S}}(\mathcal{A}))(a_{j \dots j} - r_j^{\Delta^S}(\mathcal{A})) \geq r_i^{\Delta^S}(\mathcal{A})r_j^{\overline{\Delta^S}}(\mathcal{A})$$

or $a_{i \dots i} \geq r_i(\mathcal{A})$;

- (iv) For any $i \in \bar{S}, j \in S$,

$$(a_{i \dots i} - r_i^{\overline{\Delta^S}}(\mathcal{A}))(a_{j \dots j} - r_j^{\Delta^S}(\mathcal{A})) \geq r_i^{\Delta^S}(\mathcal{A})r_j^{\overline{\Delta^S}}(\mathcal{A})$$

or $a_{i \dots i} \geq r_i(\mathcal{A})$, then \mathcal{A} is positive semi-definite.

The advantages of the results of Theorem 4.1 will be stressed by the following numerical example.

Example 4.1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,3]}$ be a real symmetric tensor with elements defined as follows:

$$\begin{aligned} a_{1111} &= 5.2, \quad a_{2222} = 6, \quad a_{3333} = 3.3, \quad a_{1112} = -0.1, \quad a_{1113} = 0.1, \\ a_{1122} &= -0.2, \quad a_{1123} = -0.2, \quad a_{1133} = 0, \quad a_{1222} = -0.1, \quad a_{1223} = 0.3, \\ a_{1233} &= 0.1, \quad a_{1333} = -0.2, \quad a_{2223} = 0.1, \quad a_{2233} = -0.1, \quad a_{2333} = 0.2. \end{aligned}$$

After some calculations, we conclude that the tensor \mathcal{A} can not meet the conditions of Theorem 3.2 in [20], and for any nonempty proper subset S of N , Theorem 4.2 of [25] and Theorem 7 of [22] can not be applied to determine the positive definiteness of \mathcal{A} , while we choose the nonempty proper subset S of N is $S = \{1, 2\}$, then $\bar{S} = \{3\}$, thus following results are easy to obtain

$$\begin{aligned} a_{1111} &= 5.2 > 3.7 = r_1^{\Delta^{\bar{S}}}(\mathcal{A}), \quad a_{2222} = 6 > 4.3 = r_2^{\Delta^{\bar{S}}}(\mathcal{A}), \\ a_{3333} &= 3.3 > 2.1 = r_3^{\Delta^{\bar{S}}}(\mathcal{A}), \\ a_{1111} &= 5.2 > 3.9 = r_1(\mathcal{A}), \quad a_{2222} = 6 > 4.5 = r_2(\mathcal{A}), \\ (a_{3333} - r_3^{\Delta^{\bar{S}}}(\mathcal{A}))(a_{1111} - r_1^{\Delta^{\bar{S}}}(\mathcal{A})) &= 5.04 > 4.93 = r_3^{\Delta^{\bar{S}}}(\mathcal{A})r_1^{\Delta^{\bar{S}}}(\mathcal{A}), \\ (a_{3333} - r_3^{\Delta^{\bar{S}}}(\mathcal{A}))(a_{2222} - r_2^{\Delta^{\bar{S}}}(\mathcal{A})) &= 6 > 5.95 = r_3^{\Delta^{\bar{S}}}(\mathcal{A})r_2^{\Delta^{\bar{S}}}(\mathcal{A}). \end{aligned}$$

This implies that \mathcal{A} satisfies the conditions (i)-(iv) in Theorem 4.1, thus \mathcal{A} is positive definite.

5 A new upper bound for the spectral radius of nonnegative tensors

On the basis of the results in Section 3, we establish a new upper bound for the spectral radius of nonnegative tensors in this section, and compare this bound with some known bounds derived in [10, 19, 25].

Theorem 5.1. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor with $n \geq 2$. And let S be a nonempty proper subset of N . Then

$$\rho(\mathcal{A}) \leq \eta_{\max}(\mathcal{A}) = \max\{\eta_1(\mathcal{A}), \eta_2(\mathcal{A}), \eta_3(\mathcal{A}), \eta_4(\mathcal{A})\},$$

where

$$\eta_1(\mathcal{A}) = \max_{i \in S} \{a_{i \dots i} + r_i^{\Delta^{\bar{S}}}(\mathcal{A})\}, \quad \eta_2(\mathcal{A}) = \max_{i \in \bar{S}} \{a_{i \dots i} + r_i^{\Delta^{\bar{S}}}(\mathcal{A})\},$$

and

$$\begin{aligned} \eta_3(\mathcal{A}) &= \max_{i \in S, j \in \bar{S}} \min \left\{ \frac{1}{2} \left(a_{i \dots i} + a_{j \dots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Phi_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}, \\ \eta_4(\mathcal{A}) &= \max_{i \in \bar{S}, j \in S} \min \left\{ \frac{1}{2} \left(a_{i \dots i} + a_{j \dots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}) + \Pi_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}, \end{aligned}$$

with

$$\begin{aligned} \Phi_{i,j}(\mathcal{A}) &= (a_{i \dots i} - a_{j \dots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\Delta^{\bar{S}}}(\mathcal{A}), \\ \Pi_{i,j}(\mathcal{A}) &= (a_{i \dots i} - a_{j \dots j} + r_i^{\Delta^{\bar{S}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\Delta^{\bar{S}}}(\mathcal{A}). \end{aligned}$$

Proof. Since \mathcal{A} is a nonnegative tensor, from Lemma 2.1, we see that $\rho(\mathcal{A})$ is an eigenvalue of \mathcal{A} , it follows from Theorem 3.1 that

$$\rho(\mathcal{A}) \in \Upsilon^S(\mathcal{A}) := \left(\Upsilon_{i,j}^S(\mathcal{A}) \right) \cup \left(\Upsilon_{i,j}^{\bar{S}}(\mathcal{A}) \right), \quad (27)$$

where

$$\begin{aligned} \Upsilon_{i,j}^S(\mathcal{A}) &= \left(\bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A}) \right) \cup \left(\bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right), \\ \Upsilon_{i,j}^{\bar{S}}(\mathcal{A}) &= \left(\bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A}) \right) \cup \left(\bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}_{i,j}^2(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right). \end{aligned}$$

If $\rho(\mathcal{A}) \in \bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A})$, then there exists $i_0 \in S$ such that $|\rho(\mathcal{A}) - a_{i_0 \dots i_0}| \leq r_{i_0}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})$. Moreover, by Lemma 2.2, we see that $\rho(\mathcal{A}) \geq \max_{i \in N} \{a_{i \dots i}\}$, then

$$\rho(\mathcal{A}) - a_{i_0 \dots i_0} \leq r_{i_0}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}),$$

i.e.,

$$\rho(\mathcal{A}) \leq a_{i_0 \dots i_0} + r_{i_0}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) \leq \max_{i \in S} \{a_{i \dots i} + r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})\}. \quad (28)$$

If $\rho(\mathcal{A}) \in \bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A})$, then there is one $i_1 \in \bar{S}$ such that $|\rho(\mathcal{A}) - a_{i_1 \dots i_1}| \leq r_{i_1}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})$, which implies that

$$\rho(\mathcal{A}) - a_{i_1 \dots i_1} \leq r_{i_1}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}),$$

i.e.,

$$\rho(\mathcal{A}) \leq a_{i_1 \dots i_1} + r_{i_1}^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) \leq \max_{i \in \bar{S}} \{a_{i \dots i} + r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})\}. \quad (29)$$

For the case that $\rho(\mathcal{A}) \in \bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right)$, then there exist $p \in S$ and $q \in \bar{S}$ such that

$$|\rho(\mathcal{A}) - a_{p \dots p}| \leq r_p(\mathcal{A}) \quad (30)$$

and

$$(|\rho(\mathcal{A}) - a_{p \dots p}| - r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}))(|\rho(\mathcal{A}) - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A})r_q^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}). \quad (31)$$

Combining Lemma 2.2 and (30) results in

$$\rho(\mathcal{A}) \leq a_{p \dots p} + r_p(\mathcal{A}) = R_p(\mathcal{A}). \quad (32)$$

Besides, by Lemma 2.2, we solve the quadratic Inequality (31) yields

$$\rho(\mathcal{A}) \leq \frac{1}{2} \{a_{p \dots p} + a_{q \dots q} + r_p^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) + r_q^{\Delta^{\bar{S}}}(\mathcal{A}) + \Phi_{p,q}^{\frac{1}{2}}(\mathcal{A})\}, \quad (33)$$

where $\Phi_{p,q}(\mathcal{A}) = (a_{p\dots p} - a_{q\dots q} + r_p^{\overline{\Delta^S}}(\mathcal{A}) - r_q^{\Delta^S}(\mathcal{A}))^2 + 4r_p^{\Delta^S}(\mathcal{A})r_q^{\overline{\Delta^S}}(\mathcal{A})$.

Combining (32) and (33) results in

$$\begin{aligned}\rho(\mathcal{A}) &\leq \min \left\{ \frac{1}{2} \left(a_{p\dots p} + a_{q\dots q} + r_p^{\overline{\Delta^S}}(\mathcal{A}) + r_q^{\Delta^S}(\mathcal{A}) + \Phi_{p,q}^{\frac{1}{2}}(\mathcal{A}) \right), R_p(\mathcal{A}) \right\} \\ &\leq \max_{i \in \bar{S}, j \in S} \min \left\{ \frac{1}{2} \left(a_{i\dots i} + a_{j\dots j} + r_i^{\overline{\Delta^S}}(\mathcal{A}) + r_j^{\Delta^S}(\mathcal{A}) + \Phi_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}.\end{aligned}\tag{34}$$

Furthermore, if $\rho(\mathcal{A}) \in \bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}_{i,j}^2(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right)$, then there exist $k \in \bar{S}$ and $l \in S$ such that

$$|\rho(\mathcal{A}) - a_{k\dots k}| \leq r_k(\mathcal{A})\tag{35}$$

and

$$(|\rho(\mathcal{A}) - a_{k\dots k}| - r_k^{\overline{\Delta^S}}(\mathcal{A}))(|\rho(\mathcal{A}) - a_{l\dots l}| - r_l^{\Delta^S}(\mathcal{A})) \leq r_k^{\Delta^S}(\mathcal{A})r_l^{\overline{\Delta^S}}(\mathcal{A}).\tag{36}$$

Combining Lemma 2.2 and (35) gives

$$\rho(\mathcal{A}) \leq a_{k\dots k} + r_k(\mathcal{A}) = R_k(\mathcal{A}).\tag{37}$$

By Lemma 2.2, Inequality (36) is equivalent to

$$\rho(\mathcal{A}) \leq \frac{1}{2} \{ a_{k\dots k} + a_{l\dots l} + r_k^{\overline{\Delta^S}}(\mathcal{A}) + r_l^{\Delta^S}(\mathcal{A}) + \Pi_{k,l}^{\frac{1}{2}}(\mathcal{A}) \},\tag{38}$$

where $\Pi_{k,l}(\mathcal{A}) = (a_{k\dots k} - a_{l\dots l} + r_k^{\overline{\Delta^S}}(\mathcal{A}) - r_l^{\Delta^S}(\mathcal{A}))^2 + 4r_k^{\Delta^S}(\mathcal{A})r_l^{\overline{\Delta^S}}(\mathcal{A})$. By (37) and (38), we obtain

$$\begin{aligned}\rho(\mathcal{A}) &\leq \min \left\{ \frac{1}{2} \left(a_{k\dots k} + a_{l\dots l} + r_k^{\overline{\Delta^S}}(\mathcal{A}) + r_l^{\Delta^S}(\mathcal{A}) + \Pi_{k,l}^{\frac{1}{2}}(\mathcal{A}) \right), R_k(\mathcal{A}) \right\} \\ &\leq \max_{i \in \bar{S}, j \in S} \min \left\{ \frac{1}{2} \left(a_{i\dots i} + a_{j\dots j} + r_i^{\overline{\Delta^S}}(\mathcal{A}) + r_j^{\Delta^S}(\mathcal{A}) + \Pi_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}.\end{aligned}\tag{39}$$

The conclusion follows from Inequalities (28), (29), (34) and (39). \blacksquare

Remark 5.1. As the upper bounds for $\rho(\mathcal{A})$ in Theorems 3.3 and 3.4 in [25] deduced from the eigenvalue localization sets $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}^S(\mathcal{A})$, respectively, and that in Theorem 5.1 derived from the eigenvalue localization set $\Upsilon^S(\mathcal{A})$. It follows from $\Upsilon^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ and the fact that $\omega_{\max}^S(\mathcal{A}) \leq \omega_{\max}(\mathcal{A}) \leq R_{\max}(\mathcal{A})$ (see Theorems 3.5 in [25]) that $\eta_{\max}(\mathcal{A}) \leq \omega_{\max}^S(\mathcal{A}) \leq \omega_{\max}(\mathcal{A}) \leq R_{\max}(\mathcal{A})$, we use $\omega_{\max}(\mathcal{A})$ and $\omega_{\max}^S(\mathcal{A})$ to denote the upper bounds in Theorems 3.3 and 3.4 in [25], respectively in this paper.

For some upper bounds we have showed that our bound is sharper than existing bounds. Now we take an example to show the efficiency of the new upper bounds.

Example 5.1. Consider the following nonnegative tensor

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :)] \in \mathbb{R}^{[3,3]},$$

where

$$A(1, :, :) = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}, A(2, :, :) = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A(3, :, :) = \begin{pmatrix} 15 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We compare the results derived in Theorem 5.1 with those in Lemma 5.2 of [10], Theorems 3.3 and 3.4 of [25]. Let $S = \{1, 2\}$, then $\bar{S} = \{3\}$. By Lemma 5.2 of [10], we have

$$\rho(\mathcal{A}) \leq 30.$$

By Theorems 3.3 and 3.4 of [25], we have

$$\rho(\mathcal{A}) \leq 29.2127.$$

By Theorem 13 of [19], we get

$$\rho(\mathcal{A}) \leq 20.2250.$$

By Theorem 5.1, we obtain

$$\rho(\mathcal{A}) \leq 15.6437.$$

This shows that the upper bound in Theorem 5.1 is sharper than those in Lemma 5.2 of [10] and Theorems 3.3-3.4 of [25], and better than the one in Theorem 13 of [19] in some cases.

6 A new lower bound for the minimum H -eigenvalue of weakly irreducible strong M -tensors

In this section, by applying the results of Theorem 3.1, we exhibit a new lower bound for the minimum H -eigenvalue of weakly irreducible strong M -tensors, which improves some existing ones derived in [12, 14].

Theorem 6.1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible strong M -tensor with $n \geq 2$. And let S be a nonempty proper subset of N . Then*

$$\tau(\mathcal{A}) \geq \pi_{\min}(\mathcal{A}) = \min\{\pi_1(\mathcal{A}), \pi_2(\mathcal{A}), \pi_3(\mathcal{A}), \pi_4(\mathcal{A})\},$$

where

$$\pi_1(\mathcal{A}) = \min_{i \in S} \{a_{i \dots i} - r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})\}, \quad \pi_2(\mathcal{A}) = \min_{i \in \bar{S}} \{a_{i \dots i} - r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A})\},$$

and

$$\begin{aligned} \pi_3(\mathcal{A}) &= \min_{i \in S, j \in \bar{S}} \max \left\{ \frac{1}{2} \left(a_{i \dots i} + a_{j \dots j} - r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) - \Theta_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}, \\ \pi_4(\mathcal{A}) &= \min_{i \in \bar{S}, j \in S} \max \left\{ \frac{1}{2} \left(a_{i \dots i} + a_{j \dots j} - r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) - r_j^{\Delta^{\bar{S}}}(\mathcal{A}) - \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}, \end{aligned}$$

with

$$\begin{aligned} \Theta_{i,j}(\mathcal{A}) &= (a_{i \dots i} - a_{j \dots j} - r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}), \\ \Lambda_{i,j}(\mathcal{A}) &= (a_{i \dots i} - a_{j \dots j} - r_i^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}) + r_j^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_i^{\Delta^{\bar{S}}}(\mathcal{A})r_j^{\bar{\Delta}^{\bar{S}}}(\mathcal{A}). \end{aligned}$$

Proof. Inasmuch as \mathcal{A} is a weakly irreducible strong M -tensor, from Lemma 2.3, $\tau(\mathcal{A})$ is an eigenvalue of \mathcal{A} , by Theorem 3.1, we have

$$\tau(\mathcal{A}) \in \Upsilon^S(\mathcal{A}) := \left(\Upsilon_{i,j}^S(\mathcal{A}) \right) \cup \left(\Upsilon_{i,j}^{\bar{S}}(\mathcal{A}) \right),$$

where

$$\begin{aligned}\mathfrak{r}_{i,j}^S(\mathcal{A}) &= \left(\bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A}) \right) \cup \left(\bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right), \\ \mathfrak{r}_{i,j}^{\bar{S}}(\mathcal{A}) &= \left(\bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A}) \right) \cup \left(\bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}_{i,j}^2(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right) \right).\end{aligned}$$

If $\tau(\mathcal{A}) \in \bigcup_{i \in S} \hat{\Upsilon}_i^1(\mathcal{A})$, then there exists $i_0 \in S$ such that $|\tau(\mathcal{A}) - a_{i_0 \dots i_0}| \leq \overline{r_{i_0}^{\Delta^{\bar{S}}}}(\mathcal{A})$. Besides, using Lemma 2.4, we know that $\tau(\mathcal{A}) \leq \min_{i \in N} \{a_{i \dots i}\}$, then

$$a_{i_0 \dots i_0} - \tau(\mathcal{A}) \leq \overline{r_{i_0}^{\Delta^{\bar{S}}}}(\mathcal{A}),$$

i.e.,

$$\tau(\mathcal{A}) \geq a_{i_0 \dots i_0} - \overline{r_{i_0}^{\Delta^{\bar{S}}}}(\mathcal{A}) \geq \min_{i \in S} \{a_{i \dots i} - \overline{r_i^{\Delta^{\bar{S}}}}(\mathcal{A})\}. \quad (40)$$

If $\tau(\mathcal{A}) \in \bigcup_{i \in \bar{S}} \hat{\Upsilon}_i^2(\mathcal{A})$, then there is one $i_1 \in \bar{S}$ such that $|\tau(\mathcal{A}) - a_{i_1 \dots i_1}| \leq \overline{r_{i_1}^{\Delta^S}}(\mathcal{A})$, together with Lemma 2.4 yields

$$a_{i_1 \dots i_1} - \tau(\mathcal{A}) \leq \overline{r_{i_1}^{\Delta^S}}(\mathcal{A}),$$

i.e.,

$$\tau(\mathcal{A}) \geq a_{i_1 \dots i_1} - \overline{r_{i_1}^{\Delta^S}}(\mathcal{A}) \geq \min_{i \in \bar{S}} \{a_{i \dots i} - \overline{r_i^{\Delta^S}}(\mathcal{A})\}. \quad (41)$$

For the case that $\tau(\mathcal{A}) \in \bigcup_{i \in S, j \in \bar{S}} \left(\tilde{\Upsilon}_{i,j}^1(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right)$, then there exist $p \in S$ and $q \in \bar{S}$ such that

$$|\tau(\mathcal{A}) - a_{p \dots p}| \leq r_p(\mathcal{A}) \quad (42)$$

and

$$(|\tau(\mathcal{A}) - a_{p \dots p}| - \overline{r_p^{\Delta^{\bar{S}}}}(\mathcal{A}))(|\tau(\mathcal{A}) - a_{q \dots q}| - r_q^{\Delta^{\bar{S}}}(\mathcal{A})) \leq r_p^{\Delta^{\bar{S}}}(\mathcal{A}) \overline{r_q^{\Delta^{\bar{S}}}}(\mathcal{A}). \quad (43)$$

Combining Lemma 2.4 and (42) gives

$$\tau(\mathcal{A}) \geq a_{p \dots p} - r_p(\mathcal{A}) = R_p(\mathcal{A}). \quad (44)$$

Having in mind that $\tau(\mathcal{A}) \leq \min_{i \in N} \{a_{i \dots i}\}$, it follows from (43) that

$$\tau(\mathcal{A}) \geq \frac{1}{2} \{a_{p \dots p} + a_{q \dots q} - \overline{r_p^{\Delta^{\bar{S}}}}(\mathcal{A}) - r_q^{\Delta^{\bar{S}}}(\mathcal{A}) - \Theta_{p,q}^{\frac{1}{2}}(\mathcal{A})\}, \quad (45)$$

where $\Theta_{p,q}(\mathcal{A}) = (a_{p \dots p} - a_{q \dots q} - \overline{r_p^{\Delta^{\bar{S}}}}(\mathcal{A}) + r_q^{\Delta^{\bar{S}}}(\mathcal{A}))^2 + 4r_p^{\Delta^{\bar{S}}}(\mathcal{A})\overline{r_q^{\Delta^{\bar{S}}}}(\mathcal{A})$.

Combining (44) and (45) results in

$$\begin{aligned}\tau(\mathcal{A}) &\geq \max \left\{ \frac{1}{2} \left(a_{p\dots p} + a_{q\dots q} - r_p^{\overline{\Delta^S}}(\mathcal{A}) - r_q^{\Delta^S}(\mathcal{A}) - \Theta_{p,q}^{\frac{1}{2}}(\mathcal{A}) \right), R_p(\mathcal{A}) \right\} \\ &\geq \min_{i \in S, j \in \bar{S}} \max \left\{ \frac{1}{2} \left(a_{i\dots i} + a_{j\dots j} - r_i^{\overline{\Delta^S}}(\mathcal{A}) - r_j^{\Delta^S}(\mathcal{A}) - \Theta_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}.\end{aligned}\quad (46)$$

Furthermore, if $\tau(\mathcal{A}) \in \bigcup_{i \in \bar{S}, j \in S} \left(\tilde{\Upsilon}_{i,j}^2(\mathcal{A}) \cap \Gamma_i(\mathcal{A}) \right)$, then there exist $k \in \bar{S}$ and $l \in S$ such that

$$|\tau(\mathcal{A}) - a_{k\dots k}| \leq r_k(\mathcal{A}) \quad (47)$$

and

$$(|\tau(\mathcal{A}) - a_{k\dots k}| - r_k^{\overline{\Delta^S}}(\mathcal{A}))(|\tau(\mathcal{A}) - a_{l\dots l}| - r_l^{\Delta^S}(\mathcal{A})) \leq r_k^{\Delta^S}(\mathcal{A})r_l^{\overline{\Delta^S}}(\mathcal{A}). \quad (48)$$

It follows from Lemma 2.4 and (47) that

$$\tau(\mathcal{A}) \geq a_{k\dots k} - r_k(\mathcal{A}) = R_k(\mathcal{A}). \quad (49)$$

On the other hand, by Lemma 2.4, solving $\tau(\mathcal{A})$ in Inequality (48) yields

$$\tau(\mathcal{A}) \geq \frac{1}{2} \{ a_{k\dots k} + a_{l\dots l} - r_k^{\overline{\Delta^S}}(\mathcal{A}) - r_l^{\Delta^S}(\mathcal{A}) - \Lambda_{k,l}^{\frac{1}{2}}(\mathcal{A}) \}, \quad (50)$$

where $\Lambda_{k,l}(\mathcal{A}) = (a_{k\dots k} - a_{l\dots l} - r_k^{\overline{\Delta^S}}(\mathcal{A}) + r_l^{\Delta^S}(\mathcal{A}))^2 + 4r_k^{\Delta^S}(\mathcal{A})r_l^{\overline{\Delta^S}}(\mathcal{A})$, which together with (49) gives

$$\begin{aligned}\tau(\mathcal{A}) &\geq \max \left\{ \frac{1}{2} \left(a_{k\dots k} + a_{l\dots l} - r_k^{\overline{\Delta^S}}(\mathcal{A}) - r_l^{\Delta^S}(\mathcal{A}) - \Lambda_{k,l}^{\frac{1}{2}}(\mathcal{A}) \right), R_k(\mathcal{A}) \right\} \\ &\geq \min_{i \in \bar{S}, j \in S} \max \left\{ \frac{1}{2} \left(a_{i\dots i} + a_{j\dots j} - r_i^{\overline{\Delta^S}}(\mathcal{A}) - r_j^{\Delta^S}(\mathcal{A}) - \Lambda_{i,j}^{\frac{1}{2}}(\mathcal{A}) \right), R_i(\mathcal{A}) \right\}.\end{aligned}\quad (51)$$

The results of this theorem follow from the Inequalities (40), (41), (46) and (51). This proves the theorem. \blacksquare

Remark 6.1. Inasmuch as the lower bounds for $\tau(\mathcal{A})$ in Theorem 2.2 in [12] and Theorem 6.1 derived from the eigenvalue localization sets $\mathcal{K}(\mathcal{A})$ and $\Upsilon^S(\mathcal{A})$, respectively. Using the similar technique as Theorem 3.5 of [25], we can prove that $\varrho_{\min}(\mathcal{A}) \geq R_{\min}(\mathcal{A})$, here we use $\varrho_{\min}(\mathcal{A})$ to denote the lower bound in Theorem 2.2 in [12]. Combining $\Upsilon^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ with $\varrho_{\min}(\mathcal{A}) \geq R_{\min}(\mathcal{A})$ results in $\pi_{\min}(\mathcal{A}) \geq \varrho_{\min}(\mathcal{A}) \geq R_{\min}(\mathcal{A})$, i.e., the lower bound in Theorem 6.1 is an improvement on those in Theorems 2.1-2.2 of [12].

Let us show that by a simple example as follows.

Example 6.1. Consider the following irreducible nonsingular M -tensor

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :)] \in \mathbb{R}^{[3,3]},$$

where

$$\begin{aligned} A(1, :, :) &= \begin{pmatrix} 12 & -2.2 & -0.3 \\ 0 & 0 & -2 \\ 0 & -1 & -1.5 \end{pmatrix}, \\ A(2, :, :) &= \begin{pmatrix} -0.5 & -4.8 & -8 \\ 0 & 30 & 0 \\ -1 & 0 & -0.5 \end{pmatrix}, \\ A(3, :, :) &= \begin{pmatrix} 0 & -3 & -1 \\ 0 & -1 & -3.5 \\ -1 & -3 & 15 \end{pmatrix}. \end{aligned}$$

We compare the results exhibited in Theorem 6.1 with those in Theorems 2.1-2.2 of [12] and Theorem 4.5 of [14]. Let $S = \{1, 2\}$, then $\bar{S} = \{3\}$. By Theorems 2.1-2.2 of [12], we have

$$\tau(\mathcal{A}) \geq 2.5.$$

By Theorem 4.5 of [14], we get

$$\tau(\mathcal{A}) \geq 2.74.$$

By Theorem 6.1, we obtain

$$\tau(\mathcal{A}) \geq 6.5,$$

which shows that the lower bound in Theorem 6.1 is much better than those in Theorems 2.1-2.2 of [12] and Theorem 4.5 of [14].

7 Concluding remarks

In this paper, a new S -type eigenvalue localization set for tensors is established, which is proved to be sharper than the ones in [22, 25]. As applications of this new set, checkable sufficient conditions for the positive definiteness and the positive semi-definiteness of tensors are proposed, these conditions have wider scope of applications compare with those of [22, 20, 25]. Moreover, based on the results of Theorem 3.1, we give new bounds for the spectral radius of nonnegative tensors and the minimum H -eigenvalue of weakly irreducible strong M -tensors, these bounds improve some existing ones obtained by Yang and Yang [10], Li et al. [25] and He and Huang [12]. Numerical experiments are also implemented to illustrate the advantages of these results.

However, the new S -type eigenvalue localization set and the derived bounds depend on the set S . How to choose S to make $\Upsilon^S(\mathcal{A})$ and the bounds exhibited in this paper as tight as possible is very important and interesting, while if the dimension of the tensor \mathcal{A} is large, this work is very difficult. Therefore, future work will include numerical or theoretical studies for finding the best choice for S .

References

- [1] T. Zhang and G.H. Golub, Rank-1 approximation of higher-order tensors, *SIAM J. Matrix Anal. Appl.* **23** (2001), 534-550.

- [2] M. Ng, L.Q. Qi and G.L. Zhou, Finding the largest eigenvalue of a nonnegative tensor, *SIAM J. Matrix Anal. Appl.* **31** (2009), 1090-1099.
- [3] R. Diamond, A note on the rotational superposition problem, *Acta Crystallogr* **44** (1988), 211-216.
- [4] L.Q. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symbolic Comput.* **40** (2005), no.6, 1302-1324.
- [5] L.Q. Qi, Symmetric nonnegative tensors and copositive tensors, *Linear Algebra Appl.* **439** (2013), 228-238.
- [6] L.Q. Qi, Eigenvalues and invariants of tensors, *J. Math. Anal. Appl.* **325** (2007), 1363-1377.
- [7] K.C. Chang, K.J. Pearson and T. Zhang, On eigenvalue problems of real symmetric tensors, *J. Math. Anal. Appl.* **350** (2009), 416-422.
- [8] K.C. Chang, K.J. Pearson and T. Zhang, Primitivity, the convergence of the NQZ method, and the largest eigenvalue for nonnegative tensors, *SIAM J. Matrix Anal. Appl.* **32** (2011), no.3, 806-819.
- [9] K.C. Chang, K.J. Pearson and T. Zhang, Some variational principles for Z-eigenvalues of nonnegative tensors, *Linear Algebra Appl.* **438** (2013), no.11, 4166-4182.
- [10] Y.N. Yang and Q.Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, *SIAM J. Matrix Anal. Appl.* **31** (2010), no.5, 2517-2530.
- [11] Y.N. Yang and Q.Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, *SIAM J. Matrix Anal. Appl.* **32** (2011), no.4, 1236-1250.
- [12] J. He and T.Z. Huang, Inequalities for M -tensors, *J. Inequal. Appl.* **114** (2014), 2014.
- [13] J. He and T.Z. Huang, Upper bound for the largest Z-eigenvalue of positive tensors, *Appl. Math. Lett.* **38** (2014), 110-114.
- [14] X.Z. Wang and Y.M. Wei, Bounds for eigenvalues of nonsingular H -tensor, *Electronic J. Linear Algebra* **29** (2015), 3-16.
- [15] L.X. Li and C.Q. Li, New bounds for the spectral radius for nonnegative tensors, *J. Inequal. Appl.* **166** (2015), 2015.
- [16] W. Li and M.K. Ng, Some bounds for the spectral radius of nonnegative tensors, *Numer. Math.* **130** (2015), 315-335.
- [17] W. Li, D.D. Liu and S.W. Vong, Z-eigenpair bounds for an irreducible nonnegative tensor, *Linear Algebra Appl.* **483** (2015), 182-199.
- [18] W.Y. Ding, L.Q. Qi and Y.M. Wei, M -tensors and nonsingular M -tensors, *Linear Algebra Appl.* **439** (2013), no.10, 3264-3278.
- [19] C.Q. Li, Z. Chen and Y.T. Li, A new eigenvalue inclusion set for tensors and its applications, *Linear Algebra Appl.* **481** (2015), 36-53.

- [20] C.Q. Li and Y.T. Li, An eigenvalue localization set for tensors with applications to determine the positive (semi-)definiteness of tensors, *Linear and Multilinear Algebra* (2015), <http://dx.doi.org/10.1080/03081087.2015.1049582>, in press.
- [21] C.Q. Li, F. Wang, J.X. Zhao, Y. Zhu and Y.T. Li, Criteria for the positive definiteness of real supersymmetric tensors, *J. Comput. Appl. Math.* **255** (2014), 1-14.
- [22] C.Q. Li, A.Q. Jiao and Y.T. Li, An S -type eigenvalue localization set for tensors, *Linear Algebra Appl.* **493** (2016), 469-483.
- [23] C.Q. Li, J.J. Zhou and Y.T. Li, A new Brauer-type eigenvalue localization set for tensors, *Linear and Multilinear Algebra* (2015), 10.1080/03081087.2015.1119779, in press.
- [24] C.J. Bu, Y.P. Wei, L.Z. Sun and J. Zhou, Brualdi-type eigenvalue inclusion sets of tensors, *Linear Algebra Appl.* **480** (2015), 168-175.
- [25] C.Q. Li, Y.T. Li and X. Kong, New eigenvalue inclusion sets for tensors, *Numer. Linear Algebra Appl.* **51** (2014), 39-50.
- [26] S.L. Hu, Z.H. Huang and L.Q. Qi, Strictly nonnegative tensors and nonnegative tensor partition, *Sci. China Math.* **57** (2014), no.1, 181-195.
- [27] L.P. Zhang, L.Q. Qi and G.L. Zhou, M -tensors and some applications, *SIAM J. Matrix Anal. Appl.* **35** (2014), no.2, 437-452.
- [28] J. Zhou, L.Z. Sun, Y.P. Wei and C.J. Bu, Some characterizations of M -tensors via digraphs, *Linear Algebra Appl.* **495** (2016), 190-198.
- [29] C.Q. Li, Y.Q. Wang, J.Y. Yi and Y.T. Li, Bounds for the spectral radius of nonnegative tensors, *Journal of Industrial and management optimization* **12** (2016), no.3, 975-990.
- [30] R.S. Varga, *Gersgorin and his circles*, Springer-Verlag, Berlin, (2004).